

MADPH-95-922

UTTG-13-95

hep-ph/9512320

Hadronic ψ production calculated in the NRQCD factorization formalism

Sean Fleming

Department of Physics, University of Wisconsin, Madison, WI 53706

Ivan Maksymyk

Theory Group, Department of Physics, University of Texas, Austin, TX 78712

Abstract

The NRQCD factorization formalism of Bodwin, Braaten, and Lepage prescribes how to write quarkonium production rates as a sum of products of short-distance coefficients times non-perturbative long-distance NRQCD matrix elements. We present, in the true spirit of the factorization formalism, a detailed calculation of the inclusive cross section for hadronic ψ production. We find that in addition to the well known *color-singlet* production mechanisms, there are equally important mechanisms in which the $c\bar{c}$ pair that forms the ψ is initially produced in a *color-octet* state, in either a 3S_1 , 1S_0 , 3P_0 or 3P_2 angular-momentum configuration. In our presentation, we emphasize the “matching” procedure, which allows us to determine the short-distance coefficients appearing in the factorization formula. We also point out how one

may systematically include relativistic corrections in these calculations.

I. INTRODUCTION

Bodwin, Braaten, and Lepage (BBL) have devised a rigorous factorization formalism that places the calculation of inclusive quarkonium annihilation and production on a solid theoretical foundation [1]. Their approach is based on nonrelativistic quantum chromodynamics (NRQCD) [2]. This is an effective field theory involving an expansion in derivatives, constructed so as to be equivalent to QCD for non-relativistic heavy quark-anti-quark scattering, to any desired order in the relative three-momentum of the heavy quarks. In Ref. [1], it is shown that the production rate for a quarkonium bound state H can be written using a factorization formula, which is a sum of products, having the form

$$\sigma(H) = \sum_n \frac{F_n}{m_Q^{d_n-4}} \langle 0 | \mathcal{O}_n^H | 0 \rangle .$$

The “short-distance” coefficients F_n are obtainable in perturbation theory as a series in $\alpha_s(m_Q)$, and can be determined through a calculation of the production rate of an on-shell $Q\bar{Q}$ pair. As to the NRQCD matrix elements $\langle 0 | \mathcal{O}_n^H | 0 \rangle$, they encode analytically uncalculable “long-distance” effects such as the hadronization of a $Q\bar{Q}$ pair into bound quarkonium. The index n labels color and angular-momentum quantum numbers, as well as the order in the NRQCD momentum expansion. In contrast to previous approaches, the BBL formalism establishes a framework within which it is possible to properly handle soft gluon effects and systematically incorporate relativistic corrections.

The size of any NRQCD matrix element can be estimated as some power of the small parameter v , the typical velocity of the heavy quarks in the bound state. (For charmonium, $v^2 \approx 0.3$.) Combining v -scaling estimates with knowledge of the size of the F_n (for which the order in $\alpha_s(m_Q)$ is easily determined), one can deduce the relative importance of the various terms in the factorization formula. Keeping only those terms warranted by experimental precision, one can cast each observable as a sum involving only a small number of NRQCD matrix elements. In this way, the important NRQCD matrix elements can be empirically determined by fitting a set of factorization formulæ to a body of experimental data.

In this paper we carefully present an example of a calculation in the factorization formalism. The quantity that we choose to calculate is the inclusive hadronic ψ production cross section, since it is needed for a complete theoretical description of ψ production at fixed target experiments. Using v -scaling rules alluded to above, we find that in addition to previously calculated color-singlet contributions, there exist equally important contributions involving the production of $c\bar{c}$ pairs in a color-octet state. The color-octet processes which turn out to be important in hadronic ψ production are those in which the heavy quark pair is produced in either a 3S_1 , 1S_0 , 3P_0 or 3P_2 angular-momentum state.

Taking the octet-mechanism contributions to hadronic ψ production as an illustrative example, we present a systematic procedure for determining the short-distance coefficients F_n appearing in the factorization formula. This procedure involves the calculation of rates for the production of $Q\bar{Q}$ pairs with specific color and angular-momentum quantum numbers. These production rates are expressed as an integral over \mathbf{q} , the relative three-momentum of the Q and \bar{Q} in the $Q\bar{Q}$ rest frame, with the integrand being a Taylor expansion in \mathbf{q}^2/m_c^2 . The production rates are calculated both in full QCD and in NRQCD, and then the results are “matched,” allowing a determination of the F_n .

The Feynman diagrams for the leading color-singlet hadronic ψ production subprocesses are shown in Figure 1. However, we do not calculate these leading color-singlet contributions since they have already been treated using the color-singlet-wavefunction model [3]. For calculations at leading order in v^2 , the results of this latter approach can be readily transformed into the factorization formalism language.

II. NRQCD FACTORIZATION APPROACH

A. Factorization Formula

As was stated in the introduction, the NRQCD factorization formalism prescribes a means of expressing the inclusive production rate of a heavy quarkonium meson H using a

factorization formula, which is a sum of products of the form

$$\sigma(A + B \rightarrow H + X) = \sum_n \frac{F_n}{m_Q^{d_n-4}} \langle 0 | \mathcal{O}_n^H | 0 \rangle . \quad (2.1)$$

The short-distance coefficients F_n can be calculated using Feynman diagram methods. Each F_n can be expressed as a perturbation series in $\alpha_s(m_Q)$. The appropriate scale for the process is m_Q since this is the scale that is associated with the production of a $Q\bar{Q}$ pair with small relative momentum.

The NRQCD matrix elements $\langle 0 | \mathcal{O}_n^H | 0 \rangle$ are of the form

$$\langle 0 | \mathcal{O}_n^H | 0 \rangle = \sum_X \sum_{m_J} \langle 0 | \mathcal{K}_n | H_{m_J} + X \rangle \langle H_{m_J} + X | \mathcal{K}'_n | 0 \rangle , \quad (2.2)$$

where \mathcal{K}_n and \mathcal{K}'_n are bilinear heavy-quark field operators; \mathcal{K}_n may take forms such as $\chi^\dagger \psi$, $\chi^\dagger \sigma^i \psi$, $\chi^\dagger \boldsymbol{\sigma} \cdot \vec{\mathbf{D}} \psi$, *etc.* and \mathcal{K}'_n takes forms such as $\psi^\dagger \chi$, *etc.* The symbol d_n appearing in the factorization formula is the combined mass dimensions of the operators \mathcal{K}_n and \mathcal{K}'_n . The heavy-quark field operators ψ and χ^\dagger are color-triplet columns, and are defined to annihilate a heavy quark and anti-quark, respectively. Thus, the \mathcal{K}_n annihilate $Q\bar{Q}$ pairs while the \mathcal{K}'_n create them. The insertion of a color matrix T^a in the bilinear operators \mathcal{K}_n and \mathcal{K}'_n , *e.g.* $\chi^\dagger T^a \psi$, means that the bilinears project a color-octet rather than a color-singlet $Q\bar{Q}$ state. The index n labels the various properties of the \mathcal{K}_n and \mathcal{K}'_n . These properties include: 1) the color-state of the $Q\bar{Q}$ pair projected by the operator (singlet or octet, designated by $\underline{1}$ or $\underline{8}$); 2) the angular-momentum state of the $Q\bar{Q}$ pair projected by the operator (given using the spectroscopic notation $^{2S+1}L_J$); and 3) the order of the operator in the momentum expansion in the NRQCD effective lagrangian. The order in the momentum expansion can be increased simply by inserting into \mathcal{K}_n or \mathcal{K}'_n the scalar $(-\frac{i}{2} \vec{\mathbf{D}})^2/m_c^2$, where $\vec{\mathbf{D}}$ is an $SU(3)$ covariant derivative. In Eq. (2.2), the sum over X is a symbolic reminder that we are calculating inclusive quarkonium production. In the discussion that follows, the sums over X and m_J will always be assumed.

The factorization formula Eq. (2.1) contains an arbitrary factorization scale Λ . In the matrix elements the scale Λ can be identified with the ultraviolet cutoff of the NRQCD

effective theory. Since physical results are independent of Λ , dependence on Λ in the short-distance coefficients cancels against that in the NRQCD matrix elements.

In the calculation of the factorization formula for a given quarkonium production rate, it is first necessary to determine which are the most important terms in the series. This operation entails a consideration of two issues: the order in $\alpha_s(m_Q)$ of the F_n and the order in v^2 of the $\langle 0|\mathcal{O}_n|0\rangle$, *i.e.* their “ v -scaling.” In the next subsection, we will turn to the issue of v -scaling of the NRQCD matrix elements.

B. v -scaling of NRQCD matrix elements

The most practical approach to the issue of v -scaling of NRQCD matrix elements is to define a v -scaling “baseline,” with all other matrix elements being suppressed by some relative power of v . This baseline corresponds to those matrix elements in which the bilinear operators \mathcal{K}_n and \mathcal{K}'_n project out the predominant $Q\bar{Q}$ component of the quarkonium, and in which this predominant component is in an S-wave. For charmonium, examples of such baseline matrix elements are

$$\begin{aligned}\langle 0|\mathcal{O}_1^\eta(^1S_0)|0\rangle &= \langle 0|\chi^\dagger\psi|\eta + X\rangle \langle \eta + X|\psi^\dagger\chi|0\rangle \\ \langle 0|\mathcal{O}_1^\psi(^3S_1)|0\rangle &= \langle 0|\chi^\dagger\boldsymbol{\sigma}\psi|\psi_{m_J} + X\rangle \cdot \langle \psi_{m_J} + X|\psi^\dagger\boldsymbol{\sigma}\chi|0\rangle.\end{aligned}\quad (2.3)$$

If one inserts the scalar operator $(-\frac{i}{2}\overleftrightarrow{\mathbf{D}})^2/m_c^2$ in either of the bilinear operators, the resulting matrix elements are suppressed by v^2 compared to those in Eqs. (2.3).

Next let us consider those matrix elements in which the bilinear operators project out the predominant $Q\bar{Q}$ component of the quarkonium, and in which this predominant component is in a P-wave. Examples are

$$\langle 0|\mathcal{O}_1^h(^1P_1)|0\rangle = \langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)\psi|h_{m_J} + X\rangle \cdot \langle h_{m_J} + X|\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\right)\chi|0\rangle \quad (2.4)$$

$$\langle 0|\mathcal{O}_1^{\chi_0}(^3P_0)|0\rangle = \frac{1}{3}\langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)\psi|\chi_0 + X\rangle \langle \chi_0 + X|\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\cdot\boldsymbol{\sigma}\right)\chi|0\rangle \quad (2.5)$$

$$\langle 0|\mathcal{O}_1^{\chi_1}(^3P_1)|0\rangle = \frac{1}{2}\langle 0|\chi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\times\boldsymbol{\sigma}\right)\psi|\chi_{1m_J} + X\rangle \cdot \langle \chi_{1m_J} + X|\psi^\dagger\left(-\frac{i}{2}\overleftrightarrow{\mathbf{D}}\times\boldsymbol{\sigma}\right)\chi|0\rangle \quad (2.6)$$

$$\langle 0 | \mathcal{O}_1^{\chi_2}({}^3P_2) | 0 \rangle = \langle 0 | \chi^\dagger \left(-\frac{i}{2} \overleftrightarrow{D}^{[i} \sigma^{j]} \right) \psi | \chi_{2m_J} + X \rangle \langle \chi_{2m_J} + X | \psi^\dagger \left(-\frac{i}{2} \overleftrightarrow{D}^{[i} \sigma^{j]} \right) \chi | 0 \rangle. \quad (2.7)$$

To a rough approximation, the above matrix elements, when divided by m_c^2 , are suppressed with respect to the baseline by v^2 , and of course, the insertion of $(-\frac{i}{2} \overleftrightarrow{\mathbf{D}})^2/m_c^2$ would result in further suppression by v^2 .

The preceding discussion has merely involved counting powers of momentum. There is, however, an important additional issue in the v -scaling of NRQCD matrix elements. This second issue hinges on the Fock state expansion of the quarkonium meson. The Fock state expansion of, for example, the ψ particle, in Coulomb gauge, can be thought of schematically as

$$\begin{aligned} |\psi\rangle = & A_{c\bar{c}} |c\bar{c}(\underline{1}, {}^3S_1)\rangle + A_{c\bar{c}g} |c\bar{c}(\underline{8}, {}^3P_J)g\rangle + A_{c\bar{c}gg} |c\bar{c}(\underline{8}, {}^3S_1)gg\rangle + B_{c\bar{c}gg} |c\bar{c}(\underline{1}, {}^3S_1)gg\rangle \\ & + C_{c\bar{c}gg} |c\bar{c}(\underline{8}, {}^3D_J)gg\rangle + D_{c\bar{c}gg} |c\bar{c}(\underline{1}, {}^3D_J)gg\rangle + B_{c\bar{c}g} |c\bar{c}(\underline{8}, {}^1S_0)g\rangle + \dots, \end{aligned} \quad (2.8)$$

where g represents a dynamical gluon, *i.e.* one whose effects cannot be incorporated into an instantaneous potential and whose typical momentum is $m_c v^2$. The angular-momentum quantum numbers of the $c\bar{c}$ pairs in the various Fock states are indicated in spectroscopic notation, and their color configurations are labeled by $\underline{1}$ for singlet or $\underline{8}$ for octet. We will now discuss the v -scaling of the various coefficients A , B , *etc.*

Since the state $|c\bar{c}(\underline{1}, {}^3S_1)\rangle$ is the predominant Fock state in $|\psi\rangle$, we expect that the coefficient $A_{c\bar{c}}$ is just a little less than unity, *i.e.* $A_{c\bar{c}} \sim v^0$. As to the Fock state $|c\bar{c}(\underline{8}, {}^3P_J)g\rangle$, this configuration arises when the predominant state radiates a soft dynamical gluon; such a process is mediated principally by the electric dipole operator, for which the selection rule is $L' = L \pm 1$, $S' = S$, and which involves a single power of heavy quark three-momentum; thus, the coefficient $A_{c\bar{c}g}$ is of order v^1 . The electric dipole emission of yet another gluon involves a change from the P -wave state $A_{c\bar{c}g} |c\bar{c}(\underline{8}, {}^3P_J)g\rangle$ to the S - and D -wave states $|c\bar{c}(\underline{8}, {}^3S_1)gg\rangle$, $|c\bar{c}(\underline{1}, {}^3S_1)gg\rangle$, $|c\bar{c}(\underline{8}, {}^3D_J)gg\rangle$ and $|c\bar{c}(\underline{1}, {}^3D_J)gg\rangle$; their coefficients — $A_{c\bar{c}gg}$, $B_{c\bar{c}gg}$, $C_{c\bar{c}gg}$ and $D_{c\bar{c}gg}$ — are of order v^2 . Lastly, we consider the coefficient of the state $|c\bar{c}(\underline{8}, {}^1S_0)g\rangle$. Fluctuations into this spin-singlet state from the predominant spin-triplet

state are associated with the emission of a soft gluon via a spin-flipping magnetic dipole transition; such transitions involve the gluon three-momentum ($\sim m_c v^2$) rather than the heavy quark three-momentum ($\sim m_c v^1$), and therefore the coefficient $B_{c\bar{c}g}$ is of order v^2 .

We now bring together the power-counting rules and the Fock state ideas presented above. Let us consider as an example the matrix element

$$\langle 0 | \mathcal{O}_8^\psi(^3S_1) | 0 \rangle = \langle 0 | \chi^\dagger \sigma^i T^a \psi | \psi_{m_J} + X \rangle \langle \psi_{m_J} + X | \psi^\dagger \sigma^i T^a \chi | 0 \rangle. \quad (2.9)$$

The bilinear operators project out the $|c\bar{c}(\underline{8}, ^3S_1)gg\rangle$ Fock state, whose coefficient in the expansion is of order v^2 . Thus, $\langle 0 | \mathcal{O}_8^\psi(^3S_1) | 0 \rangle$ is suppressed by roughly v^4 compared to the S -wave baseline.

As another example, consider the v -scaling of the matrix element $\langle 0 | \mathcal{O}_8^\psi(^3P_J) | 0 \rangle$. The bilinear operators contained therein project out the state $|c\bar{c}(\underline{8}, ^3P_J)g\rangle$, whose coefficient in the expansion would be of order v^1 . This Fock state suppression, combined with the derivatives inherent in the P -wave operators, gives a v -scaling of v^4 (with respect to the baseline) for $\langle 0 | \mathcal{O}_8^\psi(^3P_J) | 0 \rangle$.

The v -scaling of those NRQCD matrix elements pertinent to the calculation of hadronic ψ production are given in Table I.

C. Order in α_s of short-distance coefficients

The leading-order Feynman diagrams for the production of a $c\bar{c}$ pair in color-octet states are presented in Figures 2 and 3. The diagram in Figure 2 ($q\bar{q} \rightarrow g \rightarrow c\bar{c}$) produces a $c\bar{c}(\underline{8}, ^3S_1)$ state. The diagrams in Figure 3a (the gluon fusion process $gg \rightarrow c\bar{c}$) produce $c\bar{c}(\underline{8}, ^1S_0)$, $c\bar{c}(\underline{8}, ^3S_1)$ and $c\bar{c}(\underline{8}, ^3P_J)$ states. The diagram in Figure 3b ($gg \rightarrow g \rightarrow c\bar{c}$) produces a $c\bar{c}(\underline{8}, ^3S_1)$ state.

It is interesting to note that, at leading order in the \mathbf{q}^2/m_c^2 expansion, the amplitude for the production of $c\bar{c}(\underline{8}, ^3S_1)$ from Figures 3a cancels against the amplitude for the same quantum number production in Figure 3b, so that the total production of $c\bar{c}(\underline{8}, ^3S_1)$ from gluon-gluon collisions vanishes.

To determine the rough size of a term in the factorization formula, one attributes to it a power of $\alpha_s(m_c) \approx 0.3$ for each vertex in the Feynman diagram, and the appropriate power of $v^2 \approx 0.3$ according to its v -scaling. We summarize the $\alpha_s(m_c)$ powers associated with the leading order subprocesses in Table II.

Once the most important terms in the factorization formula have been identified, one proceeds to compute the short-distance coefficients F_n . This is done by matching perturbative full QCD and perturbative NRQCD calculations for the production rate of $c\bar{c}$ pairs with specific color and angular-momentum quantum numbers. These rates are expressed as an integral over relative momentum \mathbf{q} , with the integrand given as a Taylor expansion in \mathbf{q}^2/m_c^2

In the next sections, we calculate the production rates for the various important octet subprocesses contributing to hadronic ψ production, and outline in detail the matching procedure.

III. PRODUCTION RATE FOR $q\bar{q} \rightarrow \psi$ SUBPROCESS

We now calculate the leading-order contribution to the cross section for inclusive ψ production from the color-octet subprocess $q\bar{q} \rightarrow \psi$. This is done in three steps. First we carry out a perturbative QCD calculation of the production rate for the process $q(k)\bar{q}(\bar{k}) \rightarrow c(p)\bar{c}(\bar{p})$ with on-shell heavy quarks. The second step consists of calculating the production rate for the same process in NRQCD. The final step will be to determine the short-distance coefficient F_n by matching the QCD and NRQCD calculations.

A. Production rate in full QCD

In noncovariant conventions, the transition amplitude for the process $q(k)\bar{q}(\bar{k}) \rightarrow c(p)\bar{c}(\bar{p})$, illustrated in Figure 2, is given by

$$\mathcal{T} = \frac{g_s^2}{4\sqrt{k_0\bar{k}_0p_0\bar{p}_0}} \bar{v}(\bar{k})\gamma^\mu T_{ll}^a u(k) \frac{1}{(p+\bar{p})^2} \bar{u}_s(p)\gamma_\mu T_{m\bar{m}}^a v_{\bar{s}}(\bar{p}) , \quad (3.1)$$

where l, \bar{l}, m and \bar{m} are color indices and s and \bar{s} are heavy-quark spin indices. The matrices T^a are normalized throughout so that $\text{Tr}[T^a T^b] = \delta^{ab}/2$. It is convenient to re-express the c and \bar{c} momentum in terms of \mathbf{q} (the relative three-momentum of the quark and antiquark in the $c\bar{c}$ rest frame), and P (the total four-momentum of the quark and anti-quark in the lab frame):

$$\begin{aligned} p^\mu &= \frac{1}{2}P^\mu + \Lambda^{\mu i}(P)q^i, \\ \bar{p}^\mu &= \frac{1}{2}P^\mu - \Lambda^{\mu i}(P)q^i, \end{aligned} \quad (3.2)$$

where the Lorentz boost matrix $\Lambda^{\mu\nu}(P)$ is given by

$$\begin{aligned} \Lambda_j^0 &= \frac{1}{2E_q}P^j \\ \Lambda_j^i &= \delta^{ij} - \frac{P^i P^j}{\mathbf{P}^2} + \frac{P^0}{2E_q} \frac{P^i P^j}{\mathbf{P}^2}, \end{aligned} \quad (3.3)$$

with $E_q^2 = m_c^2 + \mathbf{q}^2$. The boost matrix has the following useful properties:

$$\begin{aligned} \Lambda_i^\mu \Lambda_j^\nu \delta_{ij} &= -g^{\mu\nu} + \frac{P^\mu P^\nu}{P^2} \\ \Lambda_i^\mu \Lambda_j^\nu g_{\mu\nu} &= -\delta_{ij}. \end{aligned} \quad (3.4)$$

The heavy quark Dirac bilinear $\bar{u}(p)\gamma^\mu v(\bar{p})$ appearing in Eq. (3.1) is now re-expressed in terms of \mathbf{q} , the Pauli two spinor ξ and antispinor η , and the Pauli matrices σ^i [4]:

$$\bar{u}_s(p)\gamma^\mu v_{\bar{s}}(\bar{p}) = \Lambda_j^\mu \left(2E_q \xi_s^\dagger \sigma^j \eta_{\bar{s}} - \frac{2}{E_q + m_c} q^j \xi_s^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} \eta_{\bar{s}} \right), \quad (3.5)$$

where $\bar{u}_s(p)u_s(p) = 2m_c$ and $\xi_s^\dagger \xi_t = \delta_{st}$. The procedure of re-expressing the manifestly Lorentz-invariant Dirac bilinear current in terms of two-spinors can be called the ‘‘reduction of Dirac bilinears.’’ Applying the reduction to Eq. (3.1) we obtain

$$\begin{aligned} \mathcal{T} &= \frac{g_s^2}{2\sqrt{k_0 \bar{k}_0}(P_0^2 - 4(\mathbf{\Lambda}_0 \cdot \mathbf{q})^2)} \bar{v}(\bar{k})\gamma^\mu T_{ll}^a u(k) \frac{1}{4E_q^2} \\ &\quad \times \Lambda_{j\mu}(P) \left(2E_q \xi_s^\dagger \sigma^j T_{m\bar{m}}^a \eta_{\bar{s}} - \frac{2}{E_q + m_c} q^j \xi_s^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} T_{m\bar{m}}^a \eta_{\bar{s}} \right). \end{aligned} \quad (3.6)$$

(In the above context, the symbol $T_{m\bar{m}}^a$ represents a number, not a matrix; in our notation, the two-spinors ξ and η do not carry a color index and are not acted upon by the T^a .)

Our strategy of placing the factor $T_{m\bar{m}}^a$ between the two-spinors is designed to make the matching procedure more transparent.) The bilinear combination $\xi_s^\dagger \sigma_j \eta_{\bar{s}}$ corresponds to a 3S_1 spin configuration of the $c\bar{c}$ pair. Working to leading order in \mathbf{q} , we keep only the 3S_1 term:

$$\mathcal{T} = \frac{g_s^2}{2\sqrt{k_0\bar{k}_0}(P_0^2 - 4(\boldsymbol{\Lambda}_0 \cdot \mathbf{q})^2)} \bar{v}(\bar{k}) \gamma^\mu T_{ll}^a u(k) \frac{1}{2E_q} \Lambda_{j\mu}(P) \xi_s^\dagger \sigma^j T_{m\bar{m}}^a \eta_{\bar{s}}. \quad (3.7)$$

Anticipating that the probability of hadronization is negligible for $|\mathbf{q}| > \Lambda_{QCD}$, we need only derive the algebraic form of \mathcal{T} for small \mathbf{q} , and therefore need only keep the first few terms in the Taylor expansion in \mathbf{q}^2/m_c^2 :

$$\mathcal{T} \approx \frac{g_s^2}{2P_0\sqrt{k_0\bar{k}_0}} \bar{v}(\bar{k}) \gamma^\mu T_{ll}^a u(k) \frac{1}{2m_c} \Lambda_{j\mu} \xi_s^\dagger \sigma^j T_{m\bar{m}}^a \eta_{\bar{s}} \left(1 + a \frac{\mathbf{q}^2}{m_c^2} + \dots \right), \quad (3.8)$$

where the constant a — which we do not bother to calculate explicitly — is the coefficient of the second term in the Taylor series expansion in \mathbf{q}^2/m_c^2 . It must be kept in mind that the factor appearing in brackets is not, at this point in the calculation, a rapidly converging series. Indeed, the upper bound on \mathbf{q} is associated with the total momentum available in the experiment, and in general this can be much greater than m_c . However, for the purposes of matching in the factorization formalism, we treat \mathcal{T} as an expansion in \mathbf{q}^2/m_c^2 , and consider only the first few terms, the probability of hadronization being non-negligible only for small \mathbf{q} .

Multiplying (3.8) by its complex conjugate, summing over final colors and spins, and averaging over initial colors and spins, we obtain

$$|\bar{\mathcal{T}}|^2 \approx \frac{g_s^2}{54(2m_c)^4} \xi_s^\dagger \boldsymbol{\sigma} T_{m\bar{m}}^a \eta_{\bar{s}} \cdot \eta_{\bar{s}}^\dagger \boldsymbol{\sigma} T_{\bar{m}m}^a \xi_s \left(1 + 2a \frac{\mathbf{q}^2}{m_c^2} + \dots \right), \quad (3.9)$$

where a sum over spins s and \bar{s} and color m and \bar{m} is assumed. Here we have made use of the fact that

$$\sum_{s\bar{s}} \xi_s^\dagger \sigma^i \eta_{\bar{s}} \cdot \eta_{\bar{s}}^\dagger \sigma^j \xi_s = \frac{\delta^{ij}}{3} \sum_{s\bar{s}} \xi_s^\dagger \boldsymbol{\sigma} \eta_{\bar{s}} \cdot \eta_{\bar{s}}^\dagger \boldsymbol{\sigma} \xi_s. \quad (3.10)$$

The expression for the cross section in terms of the transition amplitude squared is

$$\sigma = \frac{1}{F} \int \frac{d^3 \mathbf{p}}{(2\pi)^3} \int \frac{d^3 \bar{\mathbf{p}}}{(2\pi)^3} (2\pi)^4 \delta^4(P - k - \bar{k}) |\bar{T}|^2, \quad (3.11)$$

where $F = 2$ is the noncovariant flux. Changing variables from \mathbf{p} and $\bar{\mathbf{p}}$ to \mathbf{P} and \mathbf{q} , the above expression becomes

$$\begin{aligned} \sigma &= \frac{1}{F} \int \frac{d^3 \mathbf{P}}{(2\pi)^3} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \frac{P^0}{2E_q} (2\pi)^4 \delta^4(P - k - \bar{k}) |\bar{T}|^2 \\ &= \pi \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \delta(E_f - E_i) |\bar{T}|^2 \left(1 + a' \frac{\mathbf{q}^2}{m_c^2} + \dots \right). \end{aligned} \quad (3.12)$$

Inserting Eq. (3.9) into Eq. (3.12) we obtain the rate for on-shell $c\bar{c}$ production:

$$\sigma(q\bar{q} \rightarrow c\bar{c}(\underline{8}, {}^3S_1)) = \frac{\alpha_s^2 \pi^3}{54m_c^4} \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \delta(E_f - E_i) \xi_s^\dagger \boldsymbol{\sigma} T_{m\bar{m}}^a \eta_{\bar{s}} \eta_s^\dagger \boldsymbol{\sigma} T_{\bar{m}m}^a \xi_s \left(1 + a'' \frac{\mathbf{q}^2}{m_c^2} + \dots \right). \quad (3.13)$$

B. Production rate in NRQCD

The next step is to calculate the production rate for the process $q\bar{q} \rightarrow c\bar{c}(\underline{8}, {}^3S_1)$ in perturbative NRQCD. The first two terms are

$$\sigma(q\bar{q} \rightarrow c\bar{c}(\underline{8}, {}^3S_1)) = \int \frac{d^3 \mathbf{q}}{(2\pi)^3} \left(\frac{F_8({}^3S_1)}{m_c^2} \langle 0 | \mathcal{O}_8^{c\bar{c}(q)}({}^3S_1) | 0 \rangle + \frac{G_8({}^3S_1)}{m_c^4} \langle 0 | \mathcal{P}_8^{c\bar{c}(q)}({}^3S_1) | 0 \rangle + \dots \right), \quad (3.14)$$

where $F_8({}^3S_1)$ and $G_8({}^3S_1)$ are specific cases of the short-distance coefficients F_n , and where

$$\begin{aligned} \langle 0 | \mathcal{O}_8^{c\bar{c}(q)}({}^3S_1) | 0 \rangle &= \sum_{s\bar{s}m\bar{m}} \langle 0 | \chi^\dagger \sigma^i T^a \psi | c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) \rangle \\ &\quad \langle c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) | \psi^\dagger \sigma^i T^a \chi | 0 \rangle \end{aligned} \quad (3.15)$$

$$\begin{aligned} \langle 0 | \mathcal{P}_8^{c\bar{c}(q)}({}^3S_1) | 0 \rangle &= \frac{1}{2} \sum_{s\bar{s}m\bar{m}} \langle 0 | \chi^\dagger \sigma^i T^a \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \psi | c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) \rangle \\ &\quad \langle c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) | \psi^\dagger \sigma^i T^a \chi | 0 \rangle \\ &+ \frac{1}{2} \sum_{s\bar{s}m\bar{m}} \langle 0 | \chi^\dagger \sigma^i T^a \psi | c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) \rangle \\ &\quad \langle c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) | \psi^\dagger \sigma^i T^a \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \right)^2 \chi | 0 \rangle. \end{aligned} \quad (3.16)$$

Our task here is to derive explicit expressions for the $c\bar{c}(q)$ matrix elements in Eqs. (3.15) and (3.16). For practical reasons, it is worthwhile to quickly note one possible set of conventions for these calculations. Let us define the single-particle annihilation and creation operators to obey the anticommutation relation $[a(s, i, \mathbf{q}), a^\dagger(t, j, \mathbf{p})] = (2\pi)^3 \delta(\mathbf{q} - \mathbf{p}) \delta^{st} \delta^{ij}$. The annihilation operator acts according to $a(s, i, \mathbf{q})|c(t, j, \mathbf{p})\rangle = (2\pi)^3 \delta(\mathbf{q} - \mathbf{p}) \delta^{st} \delta^{ij} |0\rangle$, and the creation operator according to $a^\dagger(t, j, \mathbf{q})|0\rangle = |c(t, j, \mathbf{q})\rangle$. Single particle states are normalized so that $\langle c(s, i, \mathbf{q})|c(t, j, \mathbf{p})\rangle = (2\pi)^3 \delta(\mathbf{q} - \mathbf{p}) \delta^{st} \delta^{ij}$. Then, conceiving of the field operators ψ and χ as color-triplet vectors, we write their Fourier decompositions as

$$\begin{aligned}\psi_m(x) &= \sum_s \int \frac{d^3p}{(2\pi)^3} a(s, m, \mathbf{p}) \xi_s e^{-ip \cdot x} \\ \chi_{\bar{m}}^\dagger(x) &= \sum_{\bar{s}} \int \frac{d^3p}{(2\pi)^3} b(\bar{s}, \bar{m}, \mathbf{p}) \eta_{\bar{s}}^\dagger e^{-ip \cdot x},\end{aligned}\tag{3.17}$$

where ξ_s and $\eta_{\bar{s}}^\dagger$ are the two-spinors.

Using the conventions outlined in the preceding paragraph, we derive explicit expressions for the right-hand-side of Eq. (3.14), obtaining

$$\sigma(q\bar{q} \rightarrow c\bar{c}(\underline{8}, {}^3S_1)) = \int \frac{d^3\mathbf{q}}{(2\pi)^3} \left(\frac{F_8({}^3S_1)}{m_c^2} + \frac{G_8({}^3S_1)}{m_c^2} \frac{\mathbf{q}^2}{m_c^2} + \dots \right) \xi_s^\dagger \boldsymbol{\sigma} T_{m\bar{m}}^a \eta_{\bar{s}} \eta_{\bar{s}}^\dagger \boldsymbol{\sigma} T_{\bar{m}m}^a \xi_s.\tag{3.18}$$

C. Matching

Finally we determine the short-distance coefficient by matching the perturbative QCD result Eq. (3.13) to the perturbative NRQCD result Eq. (3.18)

$$F_8({}^3S_1) = \frac{2\alpha_s^2 \pi^3}{27m_c} \delta(s - 4m_c^2)\tag{3.19}$$

$$G_8({}^3S_1) = \frac{2a'' \alpha_s^2 \pi^3}{27m_c} \delta(s - 4m_c^2).\tag{3.20}$$

The matching procedure for on-shell scattering fixes the short-distance coefficient in the non-relativistic effective theory. However, these same coefficients apply to the calculation of the production of bound quarkonium states. This is because the non-perturbative effects

that will bind the c and \bar{c} into a charmonium meson take place over much longer distances than the separation $1/m_c$ associated with the formation of the $c\bar{c}$ pair. Thus, although the short-distance coefficient has been determined using a perturbative calculation of the production of free quarks and antiquarks, the same short-distance coefficient applies to the formation of a $c\bar{c}$ pair in a charmonium state. The nonperturbative effects involved in the formation of the boundstate are described by the matrix elements. Therefore, the inclusive cross section for $q\bar{q} \rightarrow \psi$ via the color-octet mechanism is, to leading order in v^2 ,

$$\begin{aligned}\sigma(q\bar{q} \rightarrow \psi) &= \frac{F_8(^3S_1)}{m_c^2} \langle 0 | \mathcal{O}_8^\psi(^3S_1) | 0 \rangle \\ &= \frac{2\alpha_s^2 \pi^3}{27m_c^3} \delta(s - 4m_c^2) \langle 0 | \mathcal{O}_8^\psi(^3S_1) | 0 \rangle .\end{aligned}\tag{3.21}$$

This procedure, which involves the determination of the F_n for the production of on-shell heavy quarks, correctly takes into account the effects of binding energy, as can be verified by explicit calculations in NRQED [5].

IV. CROSS SECTION FOR $gg \rightarrow \psi$ SUBPROCESS

Next we turn our attention to the process $g(g_1)g(g_2) \rightarrow c(p)\bar{c}(\bar{p})$ which, at leading order in α_s , proceeds through the Feynman diagrams in Figure 3. To determine the short-distance coefficients we follow a sequence of steps analogous to that outlined in the previous section. At this point we will abandon our habit of writing heavy quark spin and color indices explicitly.

The transition amplitude calculated from the diagrams in Figure 3a is

$$\begin{aligned}\mathcal{T}(gg \rightarrow c\bar{c}) &= \frac{-g_s^2}{4\sqrt{g_{10}g_{20}p_0\bar{p}_0}} \epsilon_\mu(g_1)\epsilon_\nu(g_2) \\ &\bar{u}(p) \left[T^a T^b \frac{\gamma^\mu(\not{p} - \not{\not{g}}_1 + m_c)\gamma^\nu}{2p \cdot g_1} + T^b T^a \frac{\gamma^\nu(-\not{p} + \not{\not{g}}_1 + m_c)\gamma^\mu}{2\bar{p} \cdot g_1} \right] v(\bar{p}) .\end{aligned}\tag{4.1}$$

The product of color matrices in Eq. (4.1) can be rewritten as

$$T^a T^b = \frac{1}{6} \delta^{ab} + \frac{1}{2} (d^{abc} + i f^{abc}) T^c,\tag{4.2}$$

where d^{abc} is a symmetric tensor and f^{abc} is an antisymmetric tensor. Since we wish to determine the short-distance coefficient for color-octet production, we will discard the $\delta^{ab}/6$ term of Eq. (4.2). Then, using the Dirac equation, Eq. (4.1) can be transformed into

$$\begin{aligned} \mathcal{T}(gg \rightarrow c\bar{c}) = & \frac{g_s^2}{16E_q^2\sqrt{g_{10}g_{20}p_0\bar{p}_0}} \left[1 - \frac{(g_1 \cdot \Lambda q)^2}{E_q^4} \right]^{-1} \epsilon_\mu(g_1)\epsilon_\nu(g_2) \\ & \bar{u}(p) \left\{ d^{abc}T^c \left[\mathcal{A}^{\mu\nu} - \frac{g_1 \cdot \Lambda q}{E_q^2} \mathcal{B}^{\mu\nu} \right] + i f^{abc}T^c \left[\mathcal{B}^{\mu\nu} - \frac{g_1 \cdot \Lambda q}{E_q^2} \mathcal{A}^{\mu\nu} \right] \right\} v(\bar{p}) , \end{aligned} \quad (4.3)$$

where

$$\mathcal{A}^{\mu\nu} = i\epsilon^{\mu\nu\alpha\beta}(g_1 - g_2)_\alpha \gamma_\beta \gamma_5 + 2[(\Lambda q)^\mu \gamma^\nu + (\Lambda q)^\nu \gamma^\mu] , \quad (4.4)$$

and

$$\mathcal{B}^{\mu\nu} = 2g_2^\mu \gamma^\nu - 2g_1^\nu \gamma^\mu + g^{\mu\nu}(\not{g}_1 - \not{g}_2) . \quad (4.5)$$

We reduce the Dirac bilinears using Eq. (3.5) and

$$\bar{u}(p)\gamma^\mu\gamma_5v(\bar{p}) = \frac{m_c}{E_q} P^\mu \xi^\dagger \eta - 2i\Lambda_j^\mu \xi^\dagger (\mathbf{q} \times \boldsymbol{\sigma})^j \eta . \quad (4.6)$$

Keeping only terms of leading order in \mathbf{q} , the resulting amplitude may be expressed in the form

$$\mathcal{T}(gg \rightarrow c\bar{c}) \approx \mathcal{T}_{CS}(gg \rightarrow c\bar{c}) + \mathcal{T}_{CAS}(gg \rightarrow c\bar{c}) , \quad (4.7)$$

where \mathcal{T}_{CS} is the color symmetric piece

$$\begin{aligned} \mathcal{T}_{CS}(gg \rightarrow c\bar{c}) = & d^{abc} \frac{g_s^2}{(2m_c)^4} \epsilon_\mu(g_1)\epsilon_\nu(g_2) \left\{ i\epsilon^{\mu\nu\alpha\beta}(g_1 - g_2)_\alpha P_\beta \xi^\dagger T^c \eta \right. \\ & + \left[2\epsilon^{\mu\nu\alpha\beta}(g_1 - g_2)_\alpha \epsilon^{ijk} \Lambda_\beta^k + 4m_c(\Lambda_i^\mu \Lambda_j^\nu + \Lambda_i^\nu \Lambda_j^\mu) \right. \\ & \left. \left. - \frac{2}{m_c}(g_1 \cdot \Lambda)_i (2g_2^\mu \Lambda_j^\nu - 2g_1^\nu \Lambda_j^\mu + g^{\mu\nu}(g_1 - g_2) \cdot \Lambda_j) \right] q^i \xi^\dagger \sigma^j T^c \eta \right\} + \dots , \end{aligned} \quad (4.8)$$

and where \mathcal{T}_{CAS} is the color antisymmetric part

$$\begin{aligned} \mathcal{T}_{CAS}(gg \rightarrow c\bar{c}) = & i f^{abc} \frac{g_s^2}{(2m_c)^3} \epsilon_\mu(g_1)\epsilon_\nu(g_2) \\ & \left[2g_2^\mu \Lambda_j^\nu - 2g_1^\nu \Lambda_j^\mu + g^{\mu\nu}(g_1 - g_2) \cdot \Lambda_j \right] \xi^\dagger \sigma^j T^c \eta + \dots . \end{aligned} \quad (4.9)$$

The dots \dots are included as a reminder that we are writing only the first term in the infinite and non-convergent series in \mathbf{q}^2/m_c^2 . This is justified since the factorization formalism requires only that we know the algebraic form of \mathcal{T} for small \mathbf{q} .

We note that at leading order in the non-relativistic expansion, the amplitude for producing $c\bar{c}(\underline{s}, {}^3S_1)$ in Figure 3b, Eq. (4.9) cancels against the contribution from the gluon-fusion graphs in Figure 3a, leaving only the pieces in Eq. (4.8).

We now turn to the task of isolating the terms in Eq. (4.8) that contribute to the production of $c\bar{c}$ pairs in the states $\underline{s}, {}^1S_0$, $\underline{s}, {}^3P_0$ and $\underline{s}, {}^3P_2$. Beginning with $\underline{s}, {}^1S_0$, we note that terms associated with the angular-momentum quantum numbers 1S_0 will have Pauli-spinor bilinears of the form $\xi^\dagger\eta$, $\mathbf{q}^2\xi^\dagger\eta$, *etc.* It is an easy matter to pick out the 1S_0 pieces in Eq. (4.8):

$$\mathcal{T}(gg \rightarrow c\bar{c}(\underline{s}, {}^1S_0)) = id^{abc} \frac{g_s^2}{(2m_c)^4} \epsilon_\mu(g_1)\epsilon_\nu(g_2) \epsilon^{\mu\nu\alpha\beta}(g_1 - g_2)_\alpha P_\beta \xi^\dagger T^c \eta (1 + \dots). \quad (4.10)$$

Squaring Eq. (4.10), and performing the sum-average over spins and colors, we obtain

$$|\bar{\mathcal{T}}(gg \rightarrow c\bar{c}(\underline{s}, {}^1S_0))|^2 = \frac{5g_s^4}{384(2m_c)^4} \xi^\dagger T^c \eta \eta^\dagger T^c \xi (1 + \dots). \quad (4.11)$$

Inserting this into the expression for the cross section given in Eq. (3.12) we obtain

$$\sigma(gg \rightarrow c\bar{c}(\underline{s}, {}^1S_0)) \approx \frac{5\pi^3\alpha_s^2}{3 \cdot 128 m_c^4} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \delta(E_f - E_i) \xi^\dagger T^c \eta \eta^\dagger T^c \xi (1 + \dots), \quad (4.12)$$

which is the result in full QCD for the production rate of an on-shell $c\bar{c}(\underline{s}, {}^1S_0)$.

The next step consists of calculating the same production rate in perturbative NRQCD. This is given by

$$\begin{aligned} \sigma(gg \rightarrow c\bar{c}(\underline{s}, {}^1S_0)) &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F_8({}^1S_0)}{m_c^2} \langle 0 | \mathcal{O}_8^{c\bar{c}(q)}({}^1S_0) | 0 \rangle + \dots \\ &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F_8({}^1S_0)}{m_c^2} \sum_{s\bar{s}m\bar{m}} \langle 0 | \chi^\dagger T^a \psi | c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) \rangle \\ &\quad \langle c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) | \psi^\dagger T^a \chi | 0 \rangle + \dots \\ &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F_8({}^1S_0)}{m_c^2} \xi^\dagger T^c \eta \eta^\dagger T^c \xi + \dots \end{aligned} \quad (4.13)$$

Matching the result from full QCD in Eq. (4.12) to that from NRQCD in Eq. (4.13), we obtain

$$F_8(^1S_0) = \frac{5\pi^3\alpha_s^2}{96m_c}\delta(s-4m_c^2), \quad (4.14)$$

from which we conclude that

$$\begin{aligned} \sigma(gg \rightarrow \psi + X) &= \frac{F_8(^1S_0)}{m_c^2} \langle 0 | \mathcal{O}_8^\psi(^1S_0) | 0 \rangle \\ &= \frac{5\pi^3\alpha_s^2}{96m_c^3} \delta(s-4m_c^2) \langle 0 | \mathcal{O}_8^\psi(^1S_0) | 0 \rangle. \end{aligned} \quad (4.15)$$

Next we isolate the individual P -wave contributions. This can be accomplished by first noting that any direct product of cartesian vectors may be written as

$$a^i b^j = \frac{1}{3} \delta^{ij} \mathbf{a} \cdot \mathbf{b} + \frac{1}{2} \epsilon^{ijk} (\mathbf{a} \times \mathbf{b})^k + a^{[i} b^{j]}, \quad (4.16)$$

where $a^{[i} b^{j]} = (a^i b^j + a^j b^i)/2 - \mathbf{a} \cdot \mathbf{b} \delta^{ij}/3$. Thus we are able to decompose the factor $q^i \xi^\dagger \sigma^j \eta$ appearing in the second term on the right-hand-side of Eq. (4.8) into a scalar component that is identified with the 3P_0 state, a vector component identified with the 3P_1 state, and a symmetric-traceless tensor component identified with the 3P_2 state. We wish to emphasize that the procedure of decomposing the amplitude into separate angular-momentum configurations, 1S_0 , 3S_1 , 3P_0 , 3P_1 , *etc.* can be carried out to any order in the \mathbf{q}^2/m_c^2 expansion. It is always possible to decompose the amplitude for the production of a heavy quark-antiquark pair into pieces of the form

$$\begin{aligned} \mathcal{T}(i \rightarrow c\bar{c}(^1S_0)) &= A \xi^\dagger \eta \left(1 + a \frac{\mathbf{q}^2}{m_c^2} + \dots \right) \\ \mathcal{T}(i \rightarrow c\bar{c}(^3S_1)) &= A^i \xi^\dagger \sigma^i \eta \left(1 + a \frac{\mathbf{q}^2}{m_c^2} + \dots \right) \\ \mathcal{T}(i \rightarrow c\bar{c}(^1P_1)) &= A^i \xi^\dagger q^i \eta \left(1 + a \frac{\mathbf{q}^2}{m_c^2} + \dots \right) \\ \mathcal{T}(i \rightarrow c\bar{c}(^3P_0)) &= A \xi^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} \eta \left(1 + a \frac{\mathbf{q}^2}{m_c^2} + \dots \right) \\ \mathcal{T}(i \rightarrow c\bar{c}(^3P_1)) &= A^i \xi^\dagger \mathbf{q} \times \boldsymbol{\sigma}^i \eta \left(1 + a \frac{\mathbf{q}^2}{m_c^2} + \dots \right) \\ \mathcal{T}(i \rightarrow c\bar{c}(^3P_2)) &= A^{ij} \xi^\dagger q^{[i} \sigma^{j]} \eta \left(1 + a \frac{\mathbf{q}^2}{m_c^2} + \dots \right) \end{aligned}$$

where the factors A , A^i , A^{ij} are independent of \mathbf{q} .

The 3P_0 contribution can be written as

$$\mathcal{T}(gg \rightarrow c\bar{c}(\underline{s}, {}^3P_0)) = d^{abc} \frac{4g_s^2}{(2m_c)^5} \epsilon_\mu(g_1)\epsilon_\nu(g_2) \left(-2g^{\mu\nu}m_c^2 + P^\mu g_1^\nu \right) \xi^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} T^c \eta + \dots \quad (4.17)$$

Squaring and performing the sum-average over spins and colors, we obtain

$$|\bar{\mathcal{T}}(gg \rightarrow c\bar{c}(\underline{s}, {}^3P_0))|^2 = \frac{5g_s^4}{96(2m_c)^6} \xi^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} T^c \eta \eta^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} T^c \xi (1 + \dots) \quad (4.18)$$

Inserting Eq. (4.18) into Eq. (3.12) we obtain

$$\sigma(gg \rightarrow c\bar{c}(\underline{s}, {}^3P_0)) = \frac{5\pi^3\alpha_s^2}{3 \cdot 128 m_c^6} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \delta(E_f - E_i) \xi^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} T^c \eta \eta^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} T^c \xi (1 + \dots) \quad (4.19)$$

which is the result from full QCD. As to the perturbative NRQCD result, this is

$$\begin{aligned} \sigma(gg \rightarrow c\bar{c}(\underline{s}, {}^3P_0)) &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F_8({}^3P_0)}{m_c^4} \langle 0 | \mathcal{O}_8^{c\bar{c}(q)}({}^3P_0) | 0 \rangle + \dots \\ &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F_8({}^3P_0)}{m_c^4} \sum_{s\bar{s}m\bar{m}} \frac{1}{3} \langle 0 | \chi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \boldsymbol{\sigma} \right) T^a \psi | c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) \rangle \\ &\quad \langle c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) | \psi^\dagger \left(-\frac{i}{2} \overleftrightarrow{\mathbf{D}} \cdot \boldsymbol{\sigma} \right) T^a \chi | 0 \rangle + \dots \\ &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F_8({}^3P_0)}{m_c^4} \frac{1}{3} \xi^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} T^c \eta \eta^\dagger \mathbf{q} \cdot \boldsymbol{\sigma} T^c \xi + \dots \end{aligned} \quad (4.20)$$

Matching Eqs. (4.19) and (4.20), we obtain

$$F_8({}^3P_0) = \frac{5\pi^3\alpha_s^2}{32m_c} \delta(s - 4m_c^2) \quad (4.21)$$

from which we conclude that

$$\begin{aligned} \sigma(gg \rightarrow \psi + X) &= \frac{F_8({}^3P_0)}{m_c^4} \langle 0 | \mathcal{O}_8^\psi({}^3P_0) | 0 \rangle \\ &= \frac{5\pi^3\alpha_s^2}{(2m_c)^5} \delta(s - 4m_c^2) \langle 0 | \mathcal{O}_8^\psi({}^3P_0) | 0 \rangle. \end{aligned} \quad (4.22)$$

The 3P_1 piece in Eq. (4.8) is found to vanish exactly.

Extracting the 3P_2 contribution from Eq. (4.8), one obtains

$$\begin{aligned}
\mathcal{T}(gg \rightarrow c\bar{c}(\underline{s}, {}^3P_2)) &= d^{abc} \frac{4g_s^2}{(2m_c)^4} \epsilon_\mu(g_1) \epsilon_\nu(g_2) \\
&\times \left\{ 2m_c \Lambda_i^\mu \Lambda_j^\nu - \frac{(g_1 \cdot \Lambda)_i}{m_c} (g_2^\mu \Lambda_j^\nu - g_1^\nu \Lambda_j^\mu + g^{\mu\nu} (g_1 \cdot \Lambda)_j) \right\} \\
&\times \xi^\dagger q^{[i} \sigma^{j]} T^c \eta + \dots
\end{aligned} \tag{4.23}$$

Anticipating that the cross section will involve an integral over $d^3\mathbf{q}$, we note the identity

$$\begin{aligned}
\int \frac{d^3\mathbf{q}}{(2\pi)^3} \xi^\dagger q^{[i} \sigma^{j]} T^c \eta \eta^\dagger q^{[m} \sigma^{n]} T^c \xi = \\
\frac{1}{5} \left(\frac{\delta^{im} \delta^{jn} + \delta^{in} \delta^{jm}}{2} - \frac{\delta^{ij} \delta^{mn}}{3} \right) \int \frac{d^3\mathbf{q}}{(2\pi)^3} \xi^\dagger q^{[l} \sigma^{p]} T^c \eta \eta^\dagger q^{[l} \sigma^{p]} T^c \xi,
\end{aligned} \tag{4.24}$$

which, upon squaring Eq. (4.23) and performing the sum-average over spins and colors allows us to write

$$\int \frac{d^3\mathbf{q}}{(2\pi)^3} |\bar{\mathcal{T}}(\underline{s}, {}^3P_2)|^2 = \frac{g_s^4}{24(2m_c)^6} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \xi^\dagger q^{[i} \sigma^{j]} T^c \eta \eta^\dagger q^{[i} \sigma^{j]} T^c \xi (1 + \dots). \tag{4.25}$$

Substituting Eq. (4.25) into Eq. (3.12), we obtain the full-QCD result

$$\sigma(gg \rightarrow c\bar{c}(\underline{s}, {}^3P_2)) = \frac{\pi^3 \alpha_s^2}{96m_c^6} \int \frac{d^3\mathbf{q}}{(2\pi)^3} \delta(E_f - E_i) \xi^\dagger q^{[i} \sigma^{j]} T^c \eta \eta^\dagger q^{[i} \sigma^{j]} T^c \xi (1 + \dots). \tag{4.26}$$

On the other hand, the expression for $\sigma(gg \rightarrow c\bar{c}(\underline{s}, {}^3P_2))$ calculated to lowest order in perturbative NRQCD is

$$\begin{aligned}
\sigma(gg \rightarrow c\bar{c}(\underline{s}, {}^3P_2)) &= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F_8({}^3P_2)}{m_c^4} \langle 0 | O_8^{c\bar{c}(q)}({}^3P_2) | 0 \rangle + \dots \\
&= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F_8({}^3P_2)}{m_c^4} \sum_{s\bar{s}m\bar{m}} \langle 0 | \chi^\dagger \left(-\frac{i}{2} \overleftrightarrow{D}^{[i} \sigma^{j]} \right) T^a \psi | c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) \rangle \\
&\quad \langle c(s, m, \mathbf{q}) \bar{c}(\bar{s}, \bar{m}, -\mathbf{q}) | \psi^\dagger \left(-\frac{i}{2} \overleftrightarrow{D}^{[i} \sigma^{j]} \right) T^a \chi | 0 \rangle + \dots \\
&= \int \frac{d^3\mathbf{q}}{(2\pi)^3} \frac{F_8({}^3P_2)}{m_c^4} \xi^\dagger q^{[i} \sigma^{j]} T^c \eta \eta^\dagger q^{[i} \sigma^{j]} T^c \xi + \dots
\end{aligned} \tag{4.27}$$

Matching Eqs. (4.26) and (4.27) we obtain

$$F_8({}^3P_2) = \frac{\pi^3 \alpha_s^2}{24m_c} \delta(s - 4m_c^2) \tag{4.28}$$

from which we conclude that

$$\begin{aligned}
\sigma(gg \rightarrow \psi + X) &= \frac{F_8(^3P_2)}{m_c^4} \langle 0 | \mathcal{O}_8^\psi(^3P_2) | 0 \rangle \\
&= \frac{4\pi^3 \alpha_s^2}{3(2m_c)^5} \delta(s - 4m_c^2) \langle 0 | \mathcal{O}_8^\psi(^3P_2) | 0 \rangle .
\end{aligned} \tag{4.29}$$

We can now write the proton-antiproton ψ -production cross section by convoluting the subprocess cross sections with the parton distribution functions. From the subprocess cross-section for $q\bar{q} \rightarrow \psi + X$, given in Eq. (3.21), we have

$$\begin{aligned}
\sigma(p\bar{p} \rightarrow \psi + X) &= \int dx_1 f_{q/p}(x_1) \int dx_2 f_{\bar{q}/p}(x_2) \\
&\quad \frac{2\alpha_s^2 \pi^3}{27m_c^3} \delta(x_1 x_2 s - 4m_c^2) \langle \mathcal{O}_8^\psi(^3S_1) \rangle + q \rightarrow \bar{q} ,
\end{aligned} \tag{4.30}$$

where $s = (P_p + P_{\bar{p}})^2$ is the center-of-mass energy squared of the colliding proton-antiproton system. From the subprocess cross-sections for $gg \rightarrow \psi + X$, given in Eq. (4.15), Eq. (4.22), and Eq. (4.29), we have

$$\begin{aligned}
\sigma(p\bar{p} \rightarrow \psi + X) &= \int dx_1 f_{g/p}(x_1) \int dx_2 f_{g/p}(x_2) \\
&\quad \frac{5\alpha_s^2 \pi^3}{12(2m_c)^3} \delta(x_1 x_2 s - 4m_c^2) \left(\langle \mathcal{O}_8^\psi(^1S_0) \rangle + \frac{3}{m_c^2} \langle \mathcal{O}_8^\psi(^3P_0) \rangle + \frac{4}{5m_c^2} \langle \mathcal{O}_8^\psi(^3P_2) \rangle \right) .
\end{aligned} \tag{4.31}$$

This expression can be further simplified using the relation

$$\langle \mathcal{O}_8^\psi(^3P_J) \rangle \approx (2J + 1) \langle \mathcal{O}_8^\psi(^3P_0) \rangle , \tag{4.32}$$

which holds to within corrections of relative order v^2 .

Finally we note a means of checking our results for the subprocess cross-sections $gg \rightarrow c\bar{c}(\underline{8}, ^{2S+1}L_J)$ given in Eqs. (4.15), (4.22) and (4.29). This check entails first noting the surprising fact that the amplitudes for $gg \rightarrow c\bar{c}(\underline{8}, ^{2S+1}L_J)$ are proportional to those for $gg \rightarrow c\bar{c}(\underline{1}, ^{2S+1}L_J)$. Thus, our results for octet production rates should be related — by an overall color-factor and a replacement of matrix elements — to results computed from the same subprocesses for η_c and χ_c production in the color-singlet-wavefunction model. These color-singlet results can be obtained from our color-octet results by first performing the replacements

$$\begin{aligned}
\langle \mathcal{O}_8^\psi(^1S_0) \rangle &\rightarrow \frac{15}{8} \langle \mathcal{O}_1^\eta(^1S_0) \rangle \\
\langle \mathcal{O}_8^\psi(^3P_0) \rangle &\rightarrow \frac{15}{8} \langle \mathcal{O}_1^{\chi_0}(^3P_0) \rangle \\
\langle \mathcal{O}_8^\psi(^3P_2) \rangle &\rightarrow \frac{15}{8} \langle \mathcal{O}_1^{\chi_2}(^3P_2) \rangle .
\end{aligned} \tag{4.33}$$

To convert these expressions into the language of the color-singlet-wavefunction model, one makes the further replacements

$$\begin{aligned}
\langle \mathcal{O}_1^\eta(^1S_0) \rangle &\rightarrow \frac{N_c}{2\pi} |R_s(0)|^2 \\
\langle \mathcal{O}_1^{\chi_J}(^3P_J) \rangle &\rightarrow \frac{3(2J+1)N_c}{2\pi} |R'_s(0)|^2 .
\end{aligned} \tag{4.34}$$

One observes that, indeed, these operations yield results which agree with those presented in Ref [6].

V. CONCLUSION

In this paper we have presented a calculation of the hadronic ψ production cross-section, carried out within the framework of the NRQCD factorization formalism of Bodwin, Braaten, and Lepage. We have explicitly shown how to put into practice the “matching” procedure, which allows a determination of the short-distance coefficients appearing in the factorization formula. By remaining loyal to, and expanding upon, the matching program briefly described in Ref [1], we have revealed the true spirit of the BBL formalism for quarkonium production. We have also demonstrated how, operationally, one may systematically include relativistic corrections.

We have obtained the following results for the leading order subprocess cross sections due to the color-octet mechanism:

$$\begin{aligned}
\sigma(q\bar{q} \rightarrow \psi) &= \frac{2\alpha_s^2\pi^3}{27m_c^3} \delta(s - 4m_c^2) \langle 0 | \mathcal{O}_8^\psi(^3S_1) | 0 \rangle \\
\sigma(gg \rightarrow \psi) &= \frac{5\alpha_s^2\pi^3}{96m_c^3} \delta(s - 4m_c^2) \left(\langle 0 | \mathcal{O}_8^\psi(^1S_0) | 0 \rangle + \frac{3}{m_c^2} \langle 0 | \mathcal{O}_8^\psi(^3P_0) | 0 \rangle + \frac{4}{5m_c^2} \langle 0 | \mathcal{O}_8^\psi(^3P_2) | 0 \rangle \right) .
\end{aligned}$$

As a check on these results, we have transformed the above expressions into color-singlet-wavefunction results for η_c and χ_c production, and have found them to be consistent with

previous work. We wish to point out that our expression for $\sigma(gg \rightarrow \psi)$ differs, by a factor of 3, from an analogous expressions obtained in Ref [7].

The formulæ summarized above are of great relevance in the parametrization of quarkonium production, and will be useful in future analyses. In particular, these expression, along with the previously calculated color-singlet result, appear in a leading order calculation of hadronic ψ production at fixed target experiments. In fact there are discrepancies between theoretical predictions and experimental results for ψ production in pion-nucleon interactions [8] that will be resolved by including, in the theoretical predictions, contributions from color-octet subprocesses [9].

Also note that the NRQCD matrix elements $\langle 0 | \mathcal{O}_8^\psi(^3S_1) | 0 \rangle$, $\langle 0 | \mathcal{O}_8^\psi(^1S_0) | 0 \rangle$, $\langle 0 | \mathcal{O}_8^\psi(^3P_0) | 0 \rangle$, and $\langle 0 | \mathcal{O}_8^\psi(^3P_2) | 0 \rangle$ that appear in our calculation, in the linear combinations summarized in the above formulæ, will appear in different linear combinations in a calculation of ψ production in hadronic colliders [10].

It must be pointed out that our result for $\sigma(gg \rightarrow \psi)$ is readily converted into an expression for the forward photoproduction rate $\sigma(\gamma g \rightarrow \psi)$, which is the subject of a work in progress [11].

Finally, we mention that while this work was in progress, there appeared an article by Peter Cho and Adam Leibovich [10] in which very similar work was presented. These authors calculated the leading order ψ production subprocess cross-sections that we present here, and we agree with their results. We must point out however that the approach to matching in [10] is quite different from ours, and that the two articles are complementary.

ACKNOWLEDGMENTS

It is our pleasure to thank Peter Lepage for helpful discussions. It is also our pleasure to thank Eric Braaten for his generosity, helpful discussions, and for providing us with the reduction formulæ in Eqs. (3.5) and (4.6). We thank Peter Cho and Adam Leibovich for pointing out some typographical errors in the original manuscript, and Patrick Labelle

for making some helpful comments. The work of I.M. was supported by the Robert A. Welch Foundation, by NSF Grant PHY 9009850, and by NSERC of Canada. The work of S.F. was supported in part by the U.S. Department of Energy under Grant No. DE-FG02-95ER40896, in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation, and in part by the National Science Foundation under Grant No. PHY94-07194.

REFERENCES

- [1] G.T. Bodwin, E. Braaten, and G.P. Lepage, Phys. Rev. **D51**, 1125 (1995).
- [2] W.E. Caswell and G.P. Lepage, Phys. Lett. **167B**, 437 (1986).
- [3] See, for example, G.A. Schuler, CERN preprint CERN-TH.7170/94 (hep-ph 9403387), and references therein.
- [4] E. Braaten and Y.-Q. Chen work in progress; S. Fleming Ph.D. thesis (unpublished).
- [5] Peter Lepage (private communication).
- [6] Gastmans and Wu, *The ubiquitous photon : helicity methods for QED and QCD* (Oxford University Press).
- [7] Wai-Keung Tang and M. Vanttinen, preprint SLAC-PUB-95-6931 (Jun 1995).
- [8] S.J. Brodsky, P. Hoyer, W.K. Tang, and M. Vanttinen, Phys. Rev. **D51**, 3332 (1995).
- [9] Sean Fleming and Ivan Maksymyk, work in progress.
- [10] Peter Cho and Adam Liebovich, preprint CALT-68-2026 (hep-ph/9511315).
- [11] James Amundson, Sean Fleming, and Ivan Maksymyk, preprint MADPH-95-914 and UTTG-10-95.

TABLES

TABLE I. The v -scaling of the NRQCD color-octet matrix elements, relative to S -wave baseline.

$\langle 0 \mathcal{O}_8^H(^{(2S+1)}L_J) 0 \rangle$	Scaling
$\langle 0 \mathcal{O}_1^\psi(^3S_1) 0 \rangle$	v^0
$\langle 0 \mathcal{O}_8^\psi(^3S_1) 0 \rangle$	v^4
$\langle 0 \mathcal{O}_8^\psi(^1S_0) 0 \rangle$	v^4
$\langle 0 \mathcal{O}_8^\psi(^3P_J) 0 \rangle / m_c^2$	v^4

TABLE II. The scaling with $\alpha_s(m_c)$ of the short-distance coefficients

$i + j \rightarrow c\bar{c}(n) + X$	Scaling
$gg \rightarrow c\bar{c}(\underline{1}, ^3S_1) + g$	$\alpha_s(m_c)^3$
$q\bar{q} \rightarrow c\bar{c}(\underline{8}, ^3S_1)$	$\alpha_s(m_c)^2$
$gg \rightarrow c\bar{c}(\underline{8}, ^1S_0, ^3P_{0,2})$	$\alpha_s(m_c)^2$

FIGURES

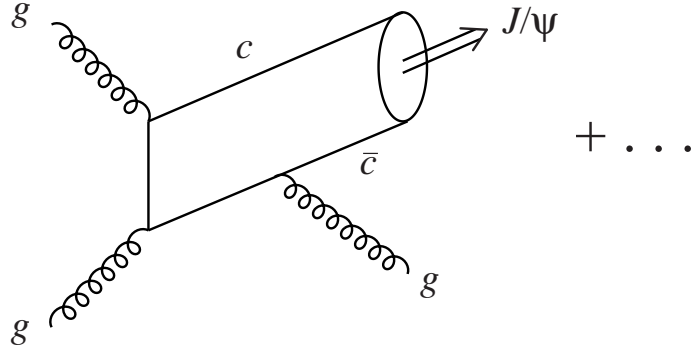


FIG. 1. Feynman diagrams for leading-order color-singlet hadronic production of charmonium.

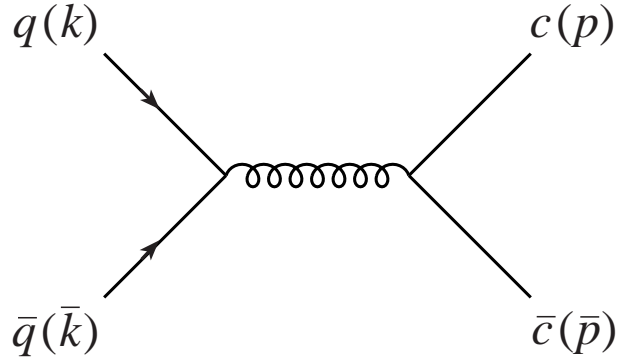


FIG. 2. Feynman diagram for the color-octet subprocess $q\bar{q} \rightarrow c\bar{c}(\underline{8}, {}^3S_1)$.

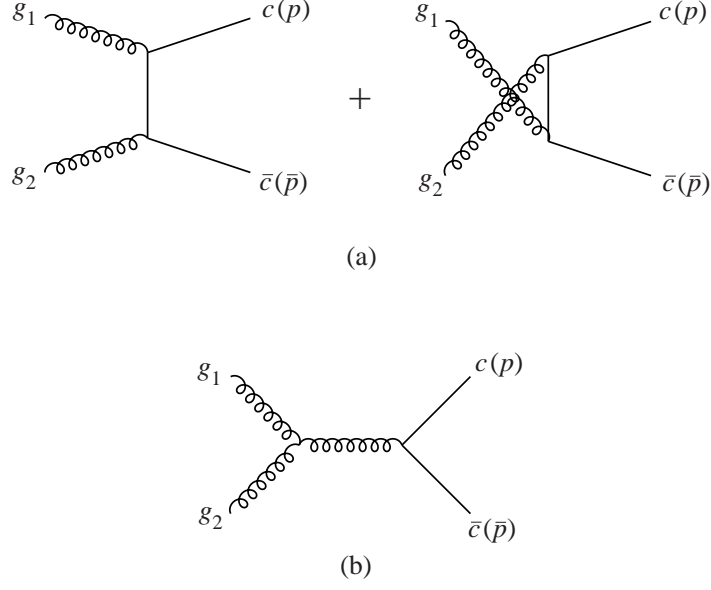


FIG. 3. Color-octet subprocesses $gg \rightarrow c\bar{c}(\underline{8}, {}^1S_0, {}^3S_1, {}^3P_{0,2})$. Figure 3a: Gluon fusion Feynman diagram, which produces $c\bar{c}(\underline{8}, {}^1S_0, {}^3S_1, {}^3P_{0,2})$. Figure 3b: Feynman diagram for $gg \rightarrow g \rightarrow c\bar{c}(\underline{8}, {}^3S_1)$. At leading order in the non-relativistic expansion, the amplitude for $c\bar{c}(\underline{8}, {}^3S_1)$ production in Figure 3a cancels against that in Figure 3b.